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The occurrence of separation in oscillatory flow

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In this short note we use the results of numerical solutions of the unsteady Navier–Stokes equations to study the occurrence of separation in unsteady flow. The structure of a flow cycle depends strongly on the Strouhal number, whose value can be classified as small, intermediate or large. In each of these regions different effects are calculated, and we give guidelines for the boundaries of these regions. In particular, we show that the small-Strouhal-number region, in which quasi-steady flow is observed, is very small and decreases rapidly as the peak Reynolds number increases.

1. Introduction

In dealing with an unsteady flow there are two fluid-dynamic parameters which characterize the nature of the flow. One is the peak Reynolds number, which in steady flow is an indicator of the relative importance of inertial and viscous effects. When the flow is unsteady the relative importance of these two effects is not constant throughout the cycle and a second parameter is used to describe the flow. This is the Strouhal number, and its value is an indication of the relative importance of unsteady and steady accelerations within the fluid. The most attractive method of developing an asymptotic theory of unsteady separation is to assume that if the Strouhal number is small enough, unsteady effects will be negligible and one can use as a basis the theory of steady separation. Asymptotic solutions of the unsteady equations have been attempted by Smith (1979), Duck (1979), Sychev (1979), Secomb (1979) and Cowley (1981). The intention of this note is to illustrate the limitations of quasi-steady theory by presenting purely numerical solutions of the Navier–Stokes equations at moderate Reynolds number.

The idea of quasi-steady flow is then that at a particular time in the flow cycle one can use the steady-flow solution calculated at the instantaneous Reynolds number. Thus the Strouhal number disappears from the leading-order term in the solution. This is best demonstrated by reference to the equations of motion. If Re_p is the peak Reynolds number and St the Strouhal number, the equation for the non-dimensional velocity $\boldsymbol{u}$ and pressure p is

$$St\frac{\partial u}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,u=-\boldsymbol{\nabla}p+\frac{1}{Re_{\mathrm{p}}}\nabla^2\boldsymbol{u}. \tag{1}$$

Let the time dependence be represented by some function $q(t)$, where for instance q might be the flux through a channel. The instantaneous Reynolds number is

$$Re(t) = Re_{\mathrm{p}}\,q(t). \tag{2}$$

Provided the Strouhal number is sufficiently small for us to neglect the unsteady term, we scale $\boldsymbol{u}$ by $q\boldsymbol{u}_s$ and p by $q^2 p_s$ to obtain

$$(\boldsymbol{u}_s \cdot \nabla)\,\boldsymbol{u}_s \approx -\nabla p_s + \frac{1}{Re(t)} \nabla^2 \boldsymbol{u}_s. \tag{3}$$

Thus $\boldsymbol{u}_s$ and p_s are given by the solution of a steady equation at the instantaneous Reynolds number $Re(t)$, and time only enters the equation in a parametric form. The neglected term $St\,\partial \boldsymbol{u}/\partial t$ should be examined further, for if $\boldsymbol{u} \approx \boldsymbol{u}_s(\boldsymbol{x}; Re(t))$ then the neglected term is actually

$$St\,\dot{q}\boldsymbol{u}_s + St\,q\frac{\partial \boldsymbol{u}_s}{\partial Re}\dot{q}\,Re_p,$$

which can be written as

$$\dot{q}\left(St\,\boldsymbol{u}_s + Re(t)\,St\frac{\partial \boldsymbol{u}_s}{\partial Re}\right).$$

In the normal course of events we have specified that $St \ll 1$, and we would expect $\partial u_s/\partial Re$ to be small. This is not true when separation occurs, for then the velocity will change rapidly as the Reynolds number varies. Hence, in a flow that separates, we would expect the criterion for quasi-steady flow to be much more restrictive than $St \ll 1$ and to involve $ReSt\,\partial u_s/\partial Re \ll 1$. Since the Reynolds number is likely to be large, the Strouhal number may need to be very small indeed. The need to combine the Strouhal number with the Reynolds number in deciding where quasi-steady theory is applicable has been observed by Cowley (1981) and Sychev (1979).

The results we report below use oscillatory flow through a furrowed channel to illustrate these ideas. Although the results are for a specific geometry we believe the structure of the flow cycle will be similar for other geometries. If $(\hat{x}, \hat{y})$ are two-dimensional Cartesian coordinates, we suppose the boundaries of a channel are given by

$$\hat{y} = \pm h\left\{1 + \tfrac{1}{2}D\left(1 - \cos\frac{2\pi\hat{x}}{L}\right)\right\}. \tag{4}$$

Non-dimensional coordinates are defined by

$$x = \frac{\hat{x}}{h}, \quad y = \frac{\hat{y}}{h}, \quad t = \Omega\hat{t}, \tag{5}$$

where $\hat{t}$ is the time. In this geometry the minimum channel gap is $2h$ and the hollow length and depth are Lh and Dh respectively. If the frequency of oscillation is Ω and the peak flux $2Uh$ the instantaneous flux is

$$\hat{q}(\hat{t}) = 2Uh \sin 2\pi\Omega\hat{t}. \tag{6}$$

Define the non-dimensional flux to be $q = 2 \sin 2\pi t$. The Reynolds and Strouhal numbers are given by

$$Re_p = \frac{Uh}{\nu}, \quad St = \frac{\Omega h}{U}, \tag{7}$$

where ν is the kinematic viscosity of the fluid. Numerical solutions of the unsteady Navier–Stokes equations have been obtained by using a finite difference scheme, and details are given in Sobey (1980).

In §2 we describe briefly the steady flow through a furrowed channel, and use those

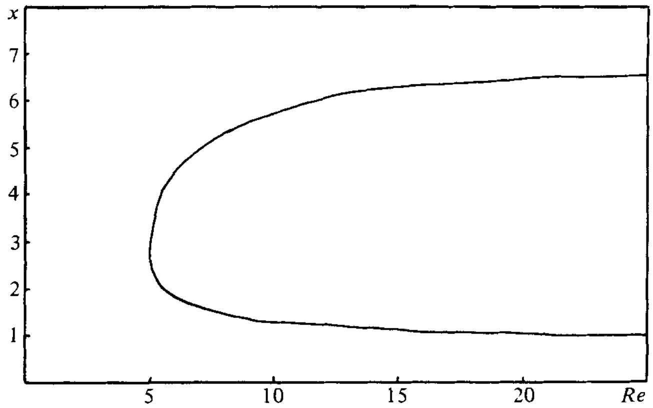

FIGURE 1. Separation envelope for steady flow through a furrowed channel; $L = 8$, $D = 2$.

results to predict the quasi-steady solution. In §3 we give details of the numerical calculations of unsteady flow, and compare those results with the quasi-steady solution. In §4 we look in further detail at the occurrence of quasi-steady flow.

2. Steady flow

There are two factors governing the motion of a fluid in a furrowed channel. There is a pressure gradient in the direction of the flow, and variation in the channel width produces additional pressure gradients on the fluid. Where the channel diverges, these gradients will oppose the driving pressure gradient and may cause separation at the walls of the channel. For a fixed geometry the occurrence of separation is indicated by a critical Reynolds number. If the actual Reynolds number of the flow is less than the critical value there will be no separation. If the critical value is exceeded then the flow separates, and further increases in the value of the Reynolds number result in the separated region extending as the vortex grows in size and strength. Variation in the geometry alters the value of the critical Reynolds number.

If the extent of the separated region is plotted against the Reynolds number an envelope is obtained, the envelope indicating the region of the channel wall that is covered by the separation bubble. This is illustrated in figure 1, where the separation envelope for a channel of length $L = 8$ and depth $D = 2$ is plotted over the range $0 < Re < 25$. All of our subsequent results in this section and in §3 are for this geometry. Thus for example at a Reynolds number of 10 the separation bubble extends over $1.2 < x < 5.7$, whilst at a Reynolds number of 30 the extent of the vortex is $1 < x < 6.6$. The separation envelope characterizes the flow, and we propose to use it in our study of unsteady flow.

We can now use the steady-flow envelope to predict that for quasi-steady flow. In unsteady flow we merely use the value of the instantaneous Reynolds number to predict the extent of the separated region. The instantaneous Reynolds number is

$$Re(t) = Re_p \sin 2\pi t, \tag{8}$$

and in figure 2 we show the separation envelope predicted by quasi-steady theory for a peak Reynolds number of 30. It can be seen that the separation envelope is now plotted against time, and only one half-cycle is necessary as the flow is periodic.

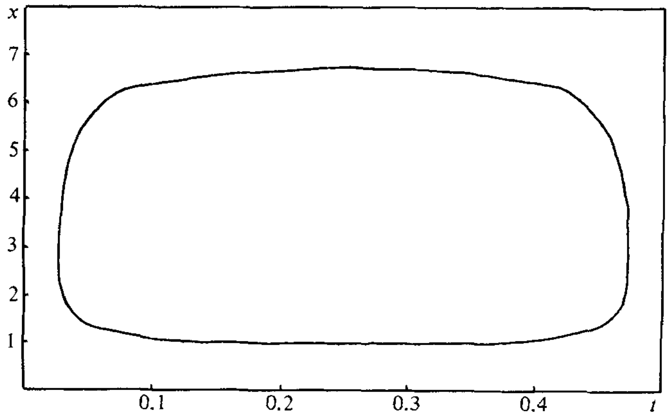

FIGURE 2. Quasi-steady prediction of separation envelope for oscillatory flow at $Re = 30$.

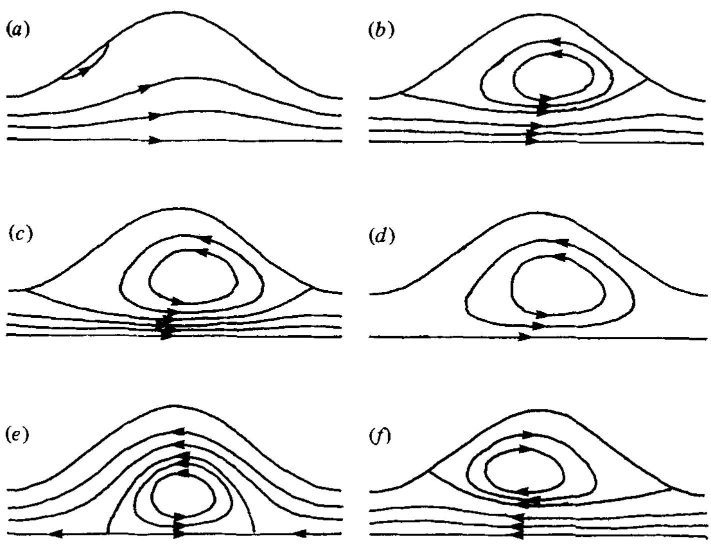

FIGURE 3. Streamlines for flow through a furrowed channel at $Re = 75$, $St = 0.01$: (*a*) $t = 0.1$; (*b*) 0.25; (*c*) 0.45; (*d*) 0.5; (*e*) 0.55; (*f*) 0.75.

3. Unsteady flow

Calculations of unsteady flows have shown a very different picture from that predicted by quasi-steady flow. Sobey (1980) has shown that a vortex may expand during periods of main flow deceleration, and those results are illustrated in figure 3. In the initial part of the cycle, as the fluid is accelerated, it streams through the channel. There are two counterbalancing pressure gradients at the entrance to the furrow. There is a pressure gradient in the direction of the mainflow which is initially proportional to the acceleration $\dot{q}$ of the fluid. There is also an opposing pressure gradient due to the channel expansion which is proportional to q^2. As $\dot{q}$ is decreasing and q^2 increasing, eventually the overall pressure gradient may cause separation, this first occurring on the upstream wall as shown in figure 3(*a*). Subsequent increase in the flux through the channel increases the size of the vortex until at peak flux, shown

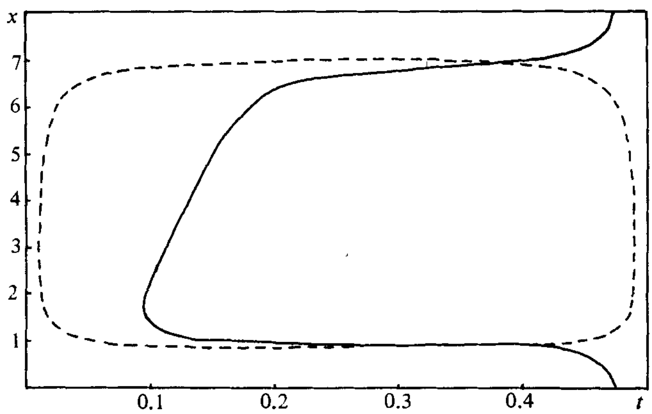

FIGURE 4. A comparison of the separation envelope calculated for $Re = 75$ and $St = 0.01$ (——) with the quasi-steady prediction (----).

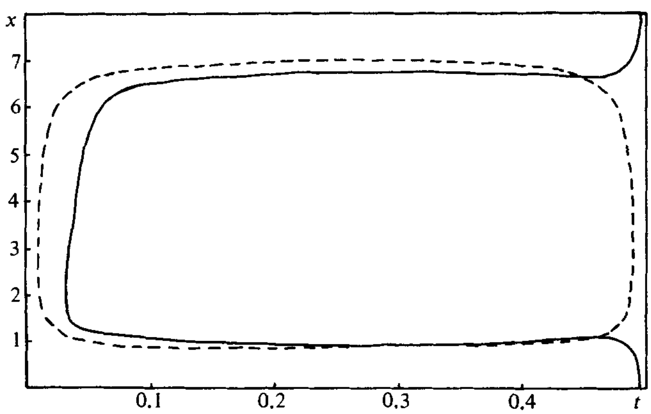

FIGURE 5. A comparison of the separation envelope calculated for $Re = 75$ and $St = 0.001$ (——) with the quasi-steady prediction (----).

in figure 3(*b*), the vortex fills most of the furrow. As the main flow decelerates, the vortex expands (figure 3*c*), and at the instant of zero flux (figure 3*d*) the vortex remains spinning on its own, filling the furrow and half the channel. Figure 3 shows half a symmetric channel and so there is another counter-rotating vortex in the other half of the channel. Reversal of the mainflow results in fluid flowing between the vortex and the wall, ejecting the vortex from the furrow (figure 3*e*). The ejected vortex is entrained into the mainstream, and separation again occurs with the formation of a counter-rotating vortex, thus repeating the process of vortex formation, growth, expansion, ejection and entrainment each half-cycle.

If we calculate the separation envelope for the flow shown in figure 3 we obtain the curve illustrated in figure 4. We have superposed onto this curve the quasi-steady prediction for the separation envelope at the peak Reynolds number of the unsteady flow. These curves are quite distinct, and they indicate fundamentally different structures for the flow cycle. It is important to note that the flow shown in figure 4 was calculated at a Strouhal number of 0.01. Sobey (1980) has already shown that at large Strouhal numbers the flow is dominated by viscous effects. The central idea of this note is that at small Strouhal numbers the flow structure becomes purely

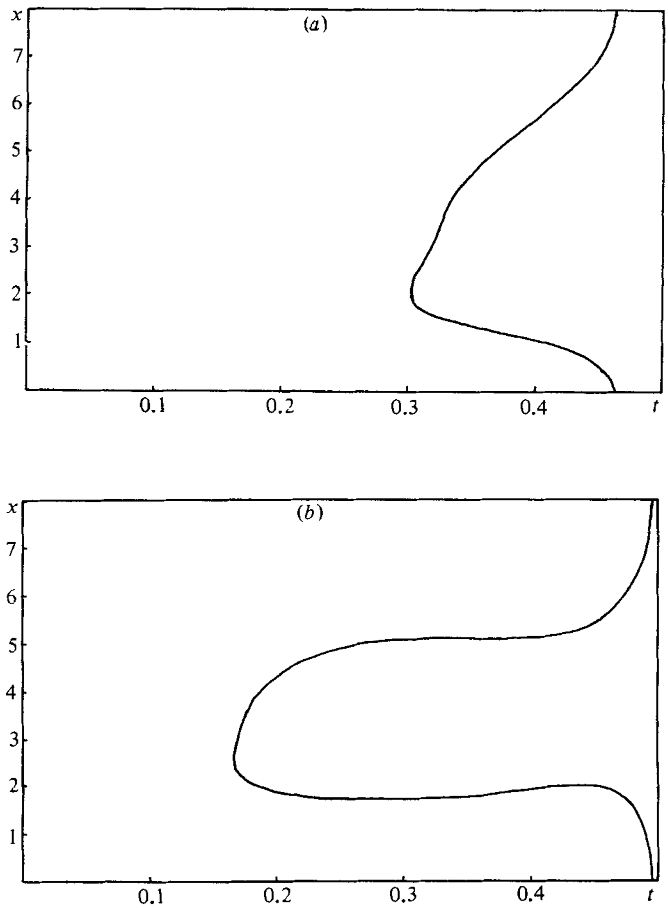

FIGURE 6 (*a*, *b*). For caption see facing page.

quasi-steady, but that between the two extremes of Strouhal number there is an intermediate region into which flows such as illustrated by figure 3 fall. To give credence to this we show in figure 5 a comparison between the separation envelopes of a flow at a Strouhal number of 0.001 and the quasi-steady prediction of figure 4. It can be seen that the degree of agreement has increased and we expect that continued reduction in the Strouhal number would increase the agreement further. The large amount of computer time required to calculate small-Strouhal-number flows makes it impractical to continue this process at this Reynolds number ($Re_p = 75$). At smaller Reynolds number quasi-steady flow is encountered at a Strouhal number for which we can carry out calculations.

At a Reynolds number of 7 we have been able to compute separation envelopes which demonstrate the three regions of Strouhal number. If the Strouhal number is large, here $St = 0.1$, then flow reversal occurs only during the deceleration. This is shown in figure 6 (*a*). This flow is dominated by viscosity and the flow reversal is akin to that in a Stokes layer formed during oscillatory flow over a flat plate. As the Strouhal number is decreased, an intermediate range is encountered, where the vortex expands during the deceleration (see figure 6*b*). If the Strouhal number is decreased still further, the vortex decreases in size during the deceleration (figure 6*c*), and at sufficiently small Strouhal numbers the separation envelope closely resembles the quasi-steady prediction (figure 6*d*). Thus three regimes exist, in which the Strouhal

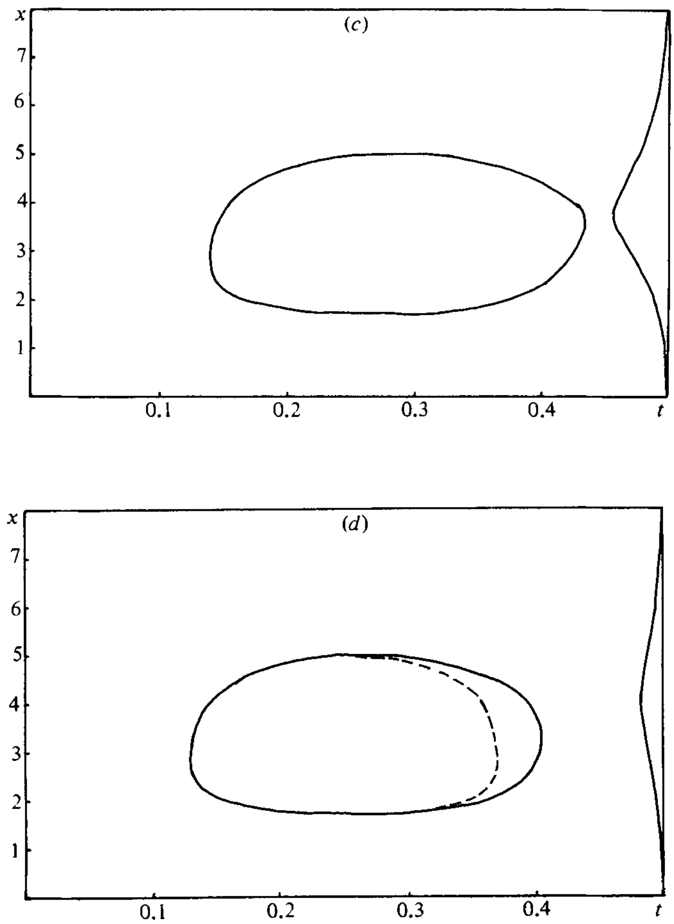

FIGURE 6. Calculated separation envelopes for $Re = 7$; (a) $St = 0.1$; (b) 0.01; (c) 0.004; (d) 0.002, ----, the envelope predicted by quasi-steady theory.

number may take small, intermediate or large values and where the behaviour of the flow is fundamentally different.

These three stages are illustrated at a Reynolds number of 30 in figure 7, where the separation envelopes are shown as the Strouhal number progresses through the three regions. The quasi-steady prediction is also shown for comparison. One should also observe that as the Reynolds number increases, here from 7 to 30 to 75, the Strouhal number for which quasi-steady theory first becomes appropriate decreases rapidly. It is this feature that we shall discuss in the next section. It is also worth noting that Sychev (1979), in discussing an asymptotic theory of unsteady separation on a flat plate, also predicts that the region in which quasi-steady theory applies decreases as the Reynolds number increases, as does Cowley (1981), who developed an asymptotic theory valid for small indentations of the wall and deduced that the Strouhal number alone being small was not enough to guarantee quasi-steady flow.

4. The occurrence of quasi-steady flow

In this section we shall define numerically the region in which quasi-steady flow can be considered a reasonable approximation to an unsteady separated flow. The most obvious criterion is that the vortex which forms during the acceleration should

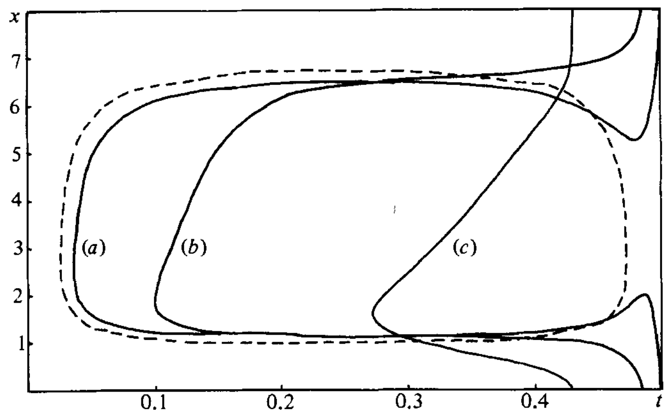

FIGURE 7. A comparison of the quasi-steady prediction (----) against envelopes calculated at $Re = 30$; (a) $St = 0.005$; (b) 0.01; (c) 0.1.

decrease in size and disappear during the deceleration. Indeed we shall insist that the vortex vanish before viscous flow reversal occurs at the end of the deceleration. Otherwise there is the possibility that the vortex will interact with the viscous flow reversal. One should also note that the viscous reversal we have mentioned is comparable with the reversal of a Stokes layer on a flat plate and results in only weak motions. Thus our criterion for quasi-steady flow is that the separation envelope should split into two parts, one showing the presence of the inertial vortex and one showing viscous flow reversal. We estimate the boundary of the quasi-steady region by first combining numerical solutions of the Stokes equations of motion with simple quasi-steady ideas and then using numerical solutions of the full Navier–Stokes equations of motion. These estimates are qualitatively identical in the region for which we have numerical solutions.

Simple quasi-steady ideas indicate that the vortex will vanish when the instantaneous Reynolds number falls below the critical value for separation of a steady flow. If the critical value is Re_c then the vortex should vanish when

$$Re_p \sin 2\pi t = Re_c, \tag{9}$$

and, provided that Re_p/Re_c is not too small,

$$t \approx 0.5 - \frac{Re_c}{2\pi Re_p}. \tag{10}$$

The next step is to estimate the time at which viscous effects will cause reverse flow. If we examine the Stokes equations for oscillatory flow in a flat channel then the stream function is given by

$$\psi = \frac{1}{2i}\left\{\frac{\sinh \theta y - y\theta \cosh \theta}{\sinh \theta - \theta \cosh \theta} e^{2\pi i t} + \text{c.c.}\right\}, \tag{11}$$

where $\theta^2 = 2\pi i Re_p St$. The wall shear is then

$$\psi_{yy}|_{y=1} = \frac{1}{2i}\left\{\frac{\theta^2 \sinh \theta \, e^{2\pi i t}}{\sinh \theta - \theta \cosh \theta} + \text{c.c.}\right\}. \tag{12}$$

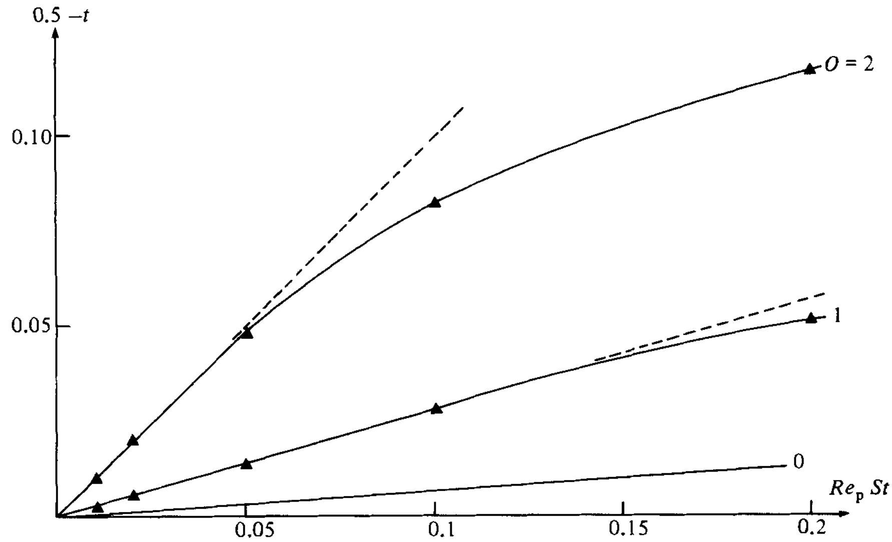

FIGURE 8. Calculated values of the time of reversal of Stokes flow for $D = 1$ and $D = 2$ as $Re\, St$ varies: ▲, calculated points; ----, lines of constant $Re\, St$.

If $Re_p\, St \ll 1$ the wall shear can be expanded for small θ,

$$\psi_{yy}|_{y=1} \sim -\frac{3}{2\mathrm{i}}\{(1+\tfrac{2}{15}\pi \mathrm{i}\, Re_p\, St)\, \mathrm{e}^{2\pi \mathrm{i} t} + \mathrm{c.c.}\}, \tag{13}$$

so that

$$\psi_{yy}|_{y=1} \sim -3\sin 2\pi(t+\tfrac{1}{15}Re_p\, St). \tag{14}$$

Thus the wall shear will reverse when $t+\frac{1}{15}Re_p\, St = 0.5$, or

$$t \approx 0.5-\tfrac{1}{15}Re_p\, St. \tag{15}$$

Thus, comparing (10) and (15), we might expect that for very shallow indentations (in which case Re_c is liable to be very large) the criterion for quasi-steady flow would be

$$St = O(Re_p^{-2}). \tag{16}$$

We have computed solutions of the Stokes equations for oscillatory flow in furrowed channels with depths $D = 1$ and $D = 2$, and our results are shown in figure 8. It can be seen that the time of viscous reversal depends linearly on $Re_p\, St$ for small values, although the coefficient varies as the depth increases. In particular, for $D = 2$ viscous reversal will occur at

$$t \sim 0.5-Re_p\, St, \tag{17}$$

so that combining this with (10) and noting that $Re_c = 5$,

$$St < 0.8\, Re_p^{-2}. \tag{18}$$

The significance of this is twofold. First it predicts an algebraic decrease in the Strouhal number as the Reynolds number increases, so that for even moderate Reynolds numbers the Strouhal number at which the flow becomes quasi-steady is liable to be very small, indeed much smaller than we might have anticipated. Secondly, we can examine numerical solutions of the full Navier–Stokes equations.

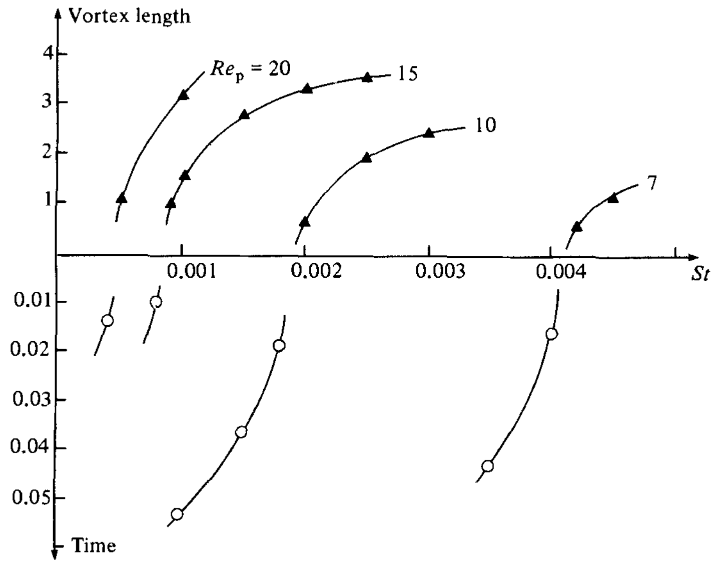

FIGURE 9. Calculation of Strouhal number for quasi-steady behaviour: ▲, calculated values of minimum vortex length during the deceleration; ⊙, calculated values of the time between disappearance of the vortex and viscous reversal.

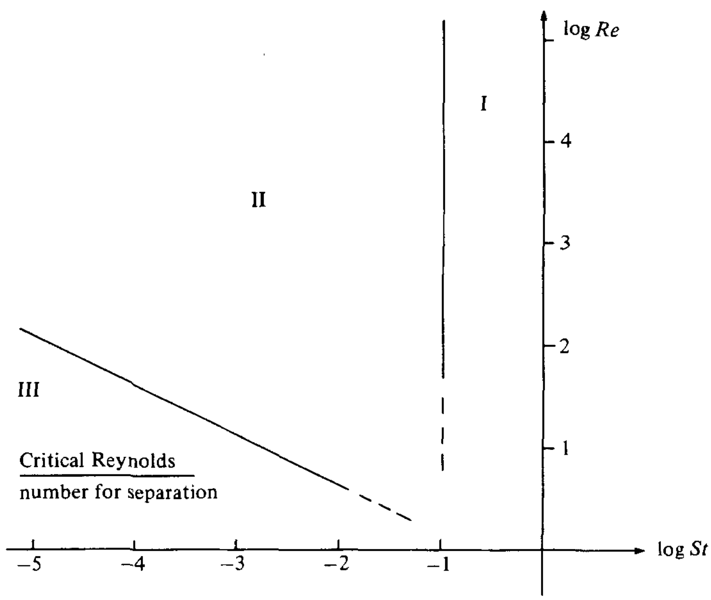

FIGURE 10. Schematic diagram of the three flow regions: I, large-Strouhal-number region; II, intermediate-Strouhal-number region; III, quasi-steady region. The critical Reynolds number is also shown.

These calculations are laborious, and for that reason we show in figure 9 a summary of our results. Above the abscissa we have plotted the minimum size of the attached region at the wall. Below the abscissa we have plotted the time between disappearance of the vortex and viscous reversal. Our criterion is obtained by calculating where both curves intercept the axis. This calculation predicts quasi-steady flow if

$$St < 0.2\, Re^{-2}. \tag{19}$$

This is in qualitative agreement with the ideas outlined above, but is smaller since one effect of unsteadiness is to delay the disappearance of the vortex.

We have sketched the three flow regions in figure 10, plotting log Re against log St. In region I, the large-Strouhal-number region, the flow is dominated by viscosity, and a reasonable approximation for the solution can be found using the unsteady Stokes equations. In region II the Strouhal number takes intermediate values and effects such as expansion of the vortex take place during the deceleration of the main flow. In region III the flow is virtually quasi-steady, but the Strouhal number is very small, and as the Reynolds number increases, the Strouhal number at the boundary of this region becomes algebraically small.

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